# Hopf-Galois Structures and Skew Braces of order $p^2q$

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# The goal and the method

## Goal

Enumerate the HGS on Galois extensions of order $p^2q$, and the skew braces of size $p^2q$ ($p$, $q$ distinct primes)

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Note also:

[Koh07] T. Kohl, *Groups of order* $4p$*, twisted wreath products and Hopf-Galois theory*, J. Algebra **314** (2007)

[Cre21] T. Crespo, *Hopf Galois structures on field extensions of degree twice an odd prime square and their associated skew left braces*, J. Algebra **565** (2021), 282-308.

[AB20a] E. Acri and M. Bonatto, *Skew braces of size* $p^2q$ *I: abelian type*, arXiv e-prints, `https://arxiv.org/abs/2004.04291` (2020).

[AB20b] E. Acri and M. Bonatto, *Skew braces of size* $p^2q$ *II: non-abelian type*, J. Algebra Appl. **21**, No.3 (2020).

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**Corollary (Greither-Pareigis (1987) and Byott (1996))**

$L/K$ Galois with group $\Gamma$. For any group $G$ with $|G| = |\Gamma|$,

$$e(\Gamma, G) = \frac{|\operatorname{Aut}(\Gamma)|}{|\operatorname{Aut}(G)|} e'(\Gamma, G).$$

$e(\Gamma, G) := |\{\text{Hopf-Galois structures on } L/K \text{ of type } G\}|;$
$e'(\Gamma, G) := |\{\text{regular subgroups of } \operatorname{Hol}(G) \text{ isomorphic to } \Gamma\}|.$

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**Guarnieri, Vendramin 2017**

$G = (G, \cdot)$ group. The following are equivalent:

1. A regular subgroup $N \leq \mathrm{Hol}(G)$
2. A group operation $\circ$ on $G$ st $(G, \cdot, \circ)$ is a *skew brace*, for $g, h, k \in G$

$$(gh) \circ k = (g \circ k)k^{-1}(h \circ k)$$

and $(G, \circ) \simeq N$

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1. A regular subgroup $N \leq \mathrm{Hol}(G)$
2. A group operation $\circ$ on $G$ st $(G, \cdot, \circ)$ is a *SB*, $(G, \circ) \simeq N$

**Byott (GV17)**

$G = (G, \cdot)$ group. There is a bijective correspondence between

- isomorphism classes of skew braces $(G, \cdot, \circ)$
- classes of regular subgroups of $\mathrm{Hol}(G)$ under conjugation by elements of $\mathrm{Aut}(G)$.

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**Caranti, Dalla Volta 2018**

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3. A map $\gamma : G \to \mathrm{Aut}(G)$ such that

$$\gamma(g^{\gamma(h)} \cdot h) = \gamma(g)\gamma(h) \qquad \text{(GFE)}$$

$\gamma$ GF on $G$ $\rightsquigarrow$
– $N = \{\gamma(g)\rho(g) : g \in G\}$
– "$\circ$" given by $g \circ h = g^{\gamma(h)} h$

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**Gamma Functions on a group $G$**

### HGS

$$e(\Gamma, G) = \frac{|\operatorname{Aut}(\Gamma)|}{|\operatorname{Aut}(G)|} e'(\Gamma, G),$$

$$e'(\Gamma, G) = |\{\gamma \text{ GF on } G : (G, \circ) \simeq \Gamma\}|$$

### SB

$(G, \cdot)$; there is a bijective correspondence between

- isomorphism classes of skew braces $(G, \cdot, \circ)$
- classes of gamma functions under "conjugation" by elements of $\operatorname{Aut}(G)$: $\gamma^{\alpha}(g) = \alpha^{-1}\gamma(g^{\alpha^{-1}})\alpha$

# What we got and how

# Groups of order $p^2q$ and their automorphism groups

| Type | Conditions | $G$ | $\mathrm{Aut}(G)$ |
|---|---|---|---|
| 1 | | $\mathcal{C}_{p^2} \times \mathcal{C}_q$ | $\mathcal{C}_{p(p-1)} \times \mathcal{C}_{q-1}$ |
| 2 | $p \mid q-1$ | $\mathcal{C}_q \rtimes_p \mathcal{C}_{p^2}$ | $\mathcal{C}_p \times \mathrm{Hol}(\mathcal{C}_q)$ |
| 3 | $p^2 \mid q-1$ | $\mathcal{C}_q \rtimes_1 \mathcal{C}_{p^2}$ | $\mathrm{Hol}(\mathcal{C}_q)$ |
| 4 | $q \mid p-1$ | $\mathcal{C}_{p^2} \rtimes \mathcal{C}_q$ | $\mathrm{Hol}(\mathcal{C}_{p^2})$ |
| 5 | | $\mathcal{C}_p \times \mathcal{C}_p \times \mathcal{C}_q$ | $\mathrm{GL}(2,p) \times \mathcal{C}_{q-1}$ |
| 6 | $q \mid p-1$ | $\mathcal{C}_p \times (\mathcal{C}_p \rtimes \mathcal{C}_q)$ | $\mathcal{C}_{p-1} \times \mathrm{Hol}(\mathcal{C}_p)$ |
| 7 | $q \mid p-1$ | $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_S \mathcal{C}_q$ | $\mathrm{Hol}(\mathcal{C}_p \times \mathcal{C}_p)$ |
| 8 | $3 < q \mid p-1$ | $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D0} \mathcal{C}_q$ | $\mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_p)$ |
| 9 | $2 < q \mid p-1$ | $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D1} \mathcal{C}_q$ | $(\mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_p)) \rtimes \mathcal{C}_2$ |
| 10 | $2 < q \mid p+1$ | $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_C \mathcal{C}_q$ | $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes (\mathcal{C}_{p^2-1} \rtimes \mathcal{C}_2)$ |
| 11 | $p \mid q-1$ | $(\mathcal{C}_q \rtimes \mathcal{C}_p) \times \mathcal{C}_p$ | $\mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_q)$ |

$L/K$ Galois of order $p^2q$, $p > 2$ and $q$ distinct primes,
$\Gamma = \mathrm{Gal}(L/K)$. $G$ group of order $p^2q$.

### For $q \nmid p-1$, the numbers $e'(\Gamma, G)$ are:

| $\Gamma$ \ $G$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $p$ | $2pq$ | $2q$ |
| 2 | $p(p-1)$ | $2p(pq-2q+1)$ | $2q(p-1)$ |
| 3 | $p^2(p-1)$ | $2p^2q(p-1)$ | $2(p^2q-pq-q+1)$ |

| $\Gamma$ \ $G$ | 5 | 11 |
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## First approach:

**Theorem (Realizability)**

Let $G$ be a group of order $p^2q$ and $\gamma$ a GF on $G$. If $p > 2$, $G$ and $(G, \circ)$ have isomorphic Sylow $p$-subgroups.

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- there always exists a Sylow $p$-subgroup $H$ which is $\gamma(H)$-invariant;
- this corresponds to have $(H, \circ)$ isomorphic to a regular subgroup of $\mathrm{Hol}(H)$;

CS19 : $H$ cyclic ($p > 2$) $\Rightarrow$ all regular subgrps of $\mathrm{Hol}(H)$ are cyclic;

FCC12 : $H$ abelian of rank $m$ with $m < p - 1$, or $m = 2$ and $p = 3 \Rightarrow$ all the abelian subgrps of $\mathrm{Hol}(H)$ are isomorphic to $H$.

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If $\Gamma$, $G$ of order $p^2q$, $p > 2$, have non isomorphic Sylow $p$-subgroups,

$$e(\Gamma, G) = e'(\Gamma, G) = 0.$$

## Second approach:

Let $G$ be a group, $A \leq G$, and $\gamma : A \to \mathrm{Aut}(G)$ a function. $\gamma$ is a *relative gamma function* (RGF) on $A$ if it satisfies the GFE and $A$ is $\gamma(A)$-invariant.

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### Lemma (Morphisms)

$G$ finite group, $A \leq G$ and $\gamma : A \to \mathrm{Aut}(G)$ a function such that $A$ is $\gamma(A)$-invariant. Any two of the following conditions imply the third one:

- $\gamma([A, \gamma(A)]) = \{1\}$. $([x, \gamma(y)] = x^{-1}x^{\gamma(y)}, x, y \in A)$
- $\gamma : A \to \mathrm{Aut}(G)$ is a morphism.
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## Third approach:

**Proposition (Lifting and restriction)**

$G$ finite group, $A, B \leq G$ such that $G = AB$.

- $\gamma$ GF on $G$ and $B \leq \ker(\gamma) \Rightarrow \gamma(ab) = \gamma(a^{\gamma(b)^{-1}})\gamma(b) = \gamma(a)$

$$\Rightarrow \gamma(G) = \gamma(A).$$

  If $A$ is $\gamma(A)$-invariant, then $\gamma_{|A} : A \to Aut(G)$ is a RGF on $A$ and $\ker(\gamma)$ is invariant under $\{\gamma'(a)\iota(a) : a \in A\} \leq \mathrm{Aut}(G)$.
- If $\gamma' : A \to \mathrm{Aut}(G)$ is a RGF such that
  1. $\gamma'(A \cap B) \equiv 1$,
  2. $B$ is invariant under $\{\gamma'(a)\iota(a) : a \in A\}$.

  Then $\gamma(ab) = \gamma'(a)$ is a GF on $G$, and $\ker(\gamma) = \ker(\gamma')B$.

Example: $p \mid q-1$, $G$ of type 1, $B$ $q$-Sylow. Necessarily $B \leq \ker(\gamma)$; moreover $A$, the $p$-Sylow, is characteristic $\Rightarrow \gamma \leftrightarrow \gamma_{|A}$

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## Fourth approach:

**Proposition (RGF on cyclic subgroups)**

$G$ finite group, $A = \langle a \rangle$ a cyclic subgroup of $G$ of order $p^n$ ($p$ odd). Let $\eta \in \mathrm{Aut}(G)$. The following are equivalent.

1. There is a RGF $\gamma : A \to \mathrm{Aut}(G)$ such that $\gamma(a) = \eta$.
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   - $ord(\eta) \mid p^n$.

Example: $p \mid q-1$, $G$ of type 1, $B$ $q$-Sylow. Necessarily $B \leq \ker(\gamma)$; moreover $A$, the $p$-Sylow, is characteristic $\Rightarrow \gamma \leftrightarrow \gamma_{|A}$;

$$\gamma_{|A} : A \to \mathrm{Aut}(G) = \mathcal{C}_{p(p-1)} \times \mathcal{C}_{q-1}$$

$$|\mathrm{GF}| = |\text{elements of order } \mid p^2 \text{ in } \mathrm{Aut}(G)| = \begin{cases} p^2 \text{ if } p \mid\mid q-1 \\ p^3 \text{ if } p^2 \mid q-1 \end{cases}$$

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| $\Gamma \backslash G$ | 5 | 11 |
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$$a^{\ominus 1} \circ b \circ a = a^{-\gamma(a)^{-1}\gamma(b)\gamma(a)} b^{\gamma(a)} a$$

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## Fifth approach:

Duality: $\rho(G)^{inv} = \lambda(G)$, where $inv : x \to x^{-1}$,

- the GF associated to the RRR $\rho(G)$ is $\gamma(x) = 1$
- the GF associated to the LRR $\lambda(G)$ is $\gamma(x) = \iota(x^{-1})$:

$$y^{\iota(x^{-1})\rho(x)} = xy = y^{\lambda(x)},$$

**More general:**

If $N \leq Hol(G)$ is a regular subgroup corresponding to $\gamma$, then $N^{inv}$ is another regular subgroup of $\mathrm{Hol}(G)$, which corresponds to

$$\widetilde{\gamma}(x) = \gamma(x^{-1})\iota(x^{-1}).$$

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(i) For $q \nmid p-1$:

| $\Gamma \backslash G$ | 1 | 2 | 3 |
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| 2 | $pq$ | $2p(pq-2q+1)$ | $2pq(p-1)$ |
| 3 | $pq$ | $2pq(p-1)$ | $2(p^2q-pq-q+1)$ |

| $\Gamma \backslash G$ | 5 | 11 |
|---|---|---|
| 5 | $p^2$ | $2p(p^2-1)$ |
| 11 | $p^2q$ | $2p(1+qp^2-2q)$ |

(ii) For $q \nmid p-1$and $q \mid p+1$:

| $\Gamma \backslash G$ | 5 | 10 |
|---|---|---|
| 5 | $p^2$ | $p(p-1)(q-1)$ |
| 10 | $p^2$ | $2+2p^2(q-3)-p^3+p^4$ |

(iii) For $q \mid p-1$:

| $\Gamma \backslash G$ | 1 | 4 |
|---|---|---|
| 1 | $p$ | $2p(q-1)$ |
| 4 | $p^2$ | $2(p^2q-2p^2+1)$ |

If $q=2$,

| $\Gamma \backslash G$ | 5 | 6 | 7 |
|---|---|---|---|
| 5 | $p^2$ | $2p(p+1)$ | $p(3p+1)$ |
| 6 | $p^2$ | $2p(p+1)$ | $p(3p+1)$ |
| 7 | $p^2$ | $2p^2(p+1)$ | $2+p(p+1)(2p-1)$ |

If $q = 3$,

| $\Gamma \backslash G$ | 5 | 6 | 7 | 9 |
|---|---|---|---|---|
| 5 | $p^2$ | $4p(p+1)$ | $2p(3p+1)$ | $4p(p+1)$ |
| 6 | $p$ | $2p(p+3)$ | $4p(p+1)$ | $p(3p+5)$ |
| 7 | $p^2$ | $2p^2(p+1)^2$ | $2+p^2(2p^2+3p+2)$ | $p(p+1)^3$ |
| 9 | $p^2(2p-1)$ | $4p(p^2+1)$ | $2(2p^3+3p^2-2p+1)$ | $2+2p+p^3(p+3)$ |

If $q > 3$,

| $\Gamma \backslash G$ | 5 | 6 |
|---|---|---|
| 5 | $p^2$ | $2p(p+1)(q-1)$ |
| 6 | $p$ | $2p(p+2q-3)$ |
| 7 | $p^2$ | $2p^2(p+1)(pq-2p+1)$ |
| 8, $G_2$ | $p^3$ | $4p(p^2+pq-3p+1)$ |
| 8, $G_k \not\simeq G_2$ | $p^2$ | $4p(p^2+pq-3p+1)$ |
| 9 | $p^2$ | $4p(p^2+pq-3p+1)$ |

| $\Gamma \backslash G$ | 7 | 9 |
|---|---|---|
| 5 | $p(3p+1)(q-1)$ | $2p(p+1)(q-1)$ |
| 6 | $4(p^2+pq-2p)$ | $p(4q+3p-7)$ |
| 7 | $2+p^2(2p^2+pq+2q-4)$ | $p(p+1)(p^2(2q-5)+2p+1)$ |
| 8, $G_2$ | $2p(p^2q-4p+pq+2)$ | $p(p^3+3p^2-14p+4pq-6)$ |
| 8, $G_k \not\simeq G_2$ | $4p(2p^2-5p+pq+2)$ | $p(p^3+5p^2-18p+4pq+8)$ |
| 9 | $2(4p^3-9p^2+2p^2q+2p+1)$ | $2+4p+p^2(p^2+5p+4q-16)$ |

| $\Gamma \backslash G8$ | $G \not\simeq G_{\pm 2}$ | $G \simeq G_{\pm 2}$, $q > 5$ | $G \simeq G_2$, $q = 5$ |
|---|---|---|---|
| 5 | $4p(p+1)(q-1)$ | $4p(p+1)(q-1)$ | $16p(p+1)$ |
| 6 | $8p(q+p-2)$ | $8p(q+p-2)$ | $8p(p+3)$ |
| 7 | $4p^2(p+1)(pq-3p+2)$ | $4p^2(p+1)(pq-3p+2)$ | $8p^2(p+1)^2$ |
| 8 | Table 2 | Table 1 | $4(1+p+3p^2(p+1))$ |
| 9 | $8p(2p^2+pq-5p+2))$ | $4p(3p^2+2pq-8p+3)$ | $16p(2p^3-2p+p+1)$ |

Table 1: $G$ and $\Gamma$ of type 8, $G \simeq G_k$ for $k = \pm 2$,

| $\Gamma$ | if $q > 7$: |
|---|---|
| $G_2$ | $2(1 + 5p + 4p^2q - 17p^2 + 7p^3)$ |
| $G_3$, $G_{\frac{3}{2}}$ | $2(7p + 4p^2q - 18p^2 + 7p^3)$ |
| $G_{-2}$ | $2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$ |
| $G_s \not\simeq G_2, G_3, G_{\frac{3}{2}}, G_{-2}$ | $8(2p + p^2q - 5p^2 + 2p^3)$ |
| $\Gamma$ | if $q = 7$: |
| $G_2$ | $2(1 + 5p + 11p^2 + 7p^3)$ |
| $G_3$ | $2(1 + 4p + 13p^2 + 6p^3)$ |

Table 2: $G$ and $\Gamma$ of type 8, $G \simeq G_k \not\simeq G_{\pm 2}$

| $\Gamma$ | if either $k$ or $k^{-1}$ is a solution of $x^2 - x - 1 = 0$: |
|---|---|
| $G_k$, $G_{1-k}$ | $2(1 + 5p + 4p^2q - 17p^2 + 7p^3)$ |
| $G_{1+k}$ | $4(3p + 2p^2q - 8p^2 + 3p^3)$ |
| $G_s \not\simeq G_k, G_{1+k}, G_{1-k}$ | $8(2p + p^2q - 5p^2 + 2p^3)$ |
| $\Gamma$ | if $k$ and $k^{-1}$ are the solutions of $x^2 + x + 1 = 0$: |
| $G_k$ | $2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$ |
| $G_{1-k}$, $G_{1-k^{-1}}$ | $2(7p + 4p^2q - 18p^2 + 7p^3)$ |
| $G_{1+k}$ | $2(1 + 4p + 4p^2q - 15p^2 + 6p^3)$ |
| $G_s \not\simeq G_k, G_{1+k}, G_{1-k}, G_{1-k^{-1}}$ | $8(2p + p^2q - 5p^2 + 2p^3)$ |
| $\Gamma$ | if $k$ and $k^{-1}$ are the solutions of $x^2 - x + 1 = 0$: |
| $G_{-k}$ | $2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$ |
| $G_{1+k}$, $G_{1+k^{-1}}$ | $2(7p + 4p^2q - 18p^2 + 7p^3)$ |
| $G_{1-k}$ | $2(1 + 4p + 4p^2q - 15p^2 + 6p^3)$ |
| $G_s \not\simeq G_{-k}, G_{1-k}, G_{1+k}, G_{1+k^{-1}}$ | $8(2p + p^2q - 5p^2 + 2p^3)$ |
| $\Gamma$ | if $k$ and $k^{-1}$ are the solutions of $x^2 + 1 = 0$: |
| $G_k$ | $4(1 + 2p + 2p^2q - 9p^2 + 4p^3)$ |
| $G_{1+k}$, $G_{1-k}$ | $4(3p + 2p^2q - 8p^2 + 3p^3)$ |
| $G_s \not\simeq G_k, G_{1+k}, G_{1-k}$ | $8(2p + p^2q - 5p^2 + 2p^3)$ |
| $\Gamma$ | if $k^2 \neq \pm k \pm 1, -1$: |
| $G_k$, $G_{-k}$ | $2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$ |
| $G_{1+k}$, $G_{1+k^{-1}}$, $G_{1-k}$, $G_{1-k^{-1}}$ | $2(7p + 4p^2q - 18p^2 + 7p^3)$ |
| $G_s \not\simeq G_{\pm k}, G_{1\pm k}, G_{1\pm k^{-1}}$ | $8(2p + p^2q - 5p^2 + 2p^3)$ |

Thank you for the attention!