### Hopf-Galois Structures and Skew Braces of order $p^2q$

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# The goal and the method

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Enumerate the HGS on Galois extensions of order  $p^2q$ , and the skew braces of size  $p^2q$  (p, q distinct primes)

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#### Note also:

- [Koh07] T. Kohl, Groups of order 4p, twisted wreath products and Hopf-Galois theory, J. Algebra 314 (2007)
- [Cre21] T. Crespo, Hopf Galois structures on field extensions of degree twice an odd prime square and their associated skew left braces, J. Algebra 565 (2021), 282-308.
- [AB20a] E. Acri and M. Bonatto, Skew braces of size p<sup>2</sup>q l: abelian type, arXiv e-prints, https://arxiv.org/abs/2004.04291 (2020).
- [AB20b] E. Acri and M. Bonatto, Skew braces of size p<sup>2</sup>q II: non-abelian type, J. Algebra Appl. 21, No.3 (2020).

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regular subgroups of the holomorph of a group G

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# regular subgroups of the holomorph of a group G

Corollary (Greither-Pareigis (1987) and Byott (1996))

L/K Galois with group  $\Gamma$ . For any group G with  $|G| = |\Gamma|$ ,

$$e(\Gamma, G) = \frac{|\operatorname{Aut}(\Gamma)|}{|\operatorname{Aut}(G)|} e'(\Gamma, G).$$

 $e(\Gamma, G) := |\{\text{Hopf-Galois structures on } L/K \text{ of type } G\}|;$  $e'(\Gamma, G) := |\{\text{regular subgroups of Hol}(G) \text{ isomorphic to } \Gamma\}|.$ 

In order to enumerate both the HGS on Galois extensions and the SB we study

# regular subgroups of the holomorph of a group G

#### Guarnieri, Vendramin 2017

 $G = (G, \cdot)$  group. The following are equivalent:

- A regular subgroup  $N \leq Hol(G)$
- ② A group operation on G st (G, ·, ○) is a skew brace, for g, h, k ∈ G

$$(gh) \circ k = (g \circ k)k^{-1}(h \circ k)$$

and  $(G,\circ)\simeq N$ 

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- 2 A group operation  $\circ$  on G st  $(G, \cdot, \circ)$  is a SB,  $(G, \circ) \simeq N$

#### Byott (GV17)

 ${\it G}=({\it G},\cdot)$  group. There is a bijective correspondence between

- isomorphism classes of skew braces  $(G, \cdot, \circ)$
- classes of regular subgroups of Hol(G) under conjugation by elements of Aut(G).

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#### Caranti, Dalla Volta 2018

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- A regular subgroup  $N \leq Hol(G)$
- 2 A group operation  $\circ$  on G st  $(G, \cdot, \circ)$  is a SB,  $(G, \circ) \simeq N$
- **3** A map  $\gamma: G \to \operatorname{Aut}(G)$  such that

$$\gamma(g^{\gamma(h)} \cdot h) = \gamma(g)\gamma(h)$$
 (GFE)

 $\gamma \text{ GF on } G \longrightarrow -N = \{\gamma(g)\rho(g) : g \in G\}$ - "  $\circ$ " given by  $g \circ h = g^{\gamma(h)}h$ 

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Gamma Functions on a group G

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Gamma Functions on a group G

# $e(\Gamma, G) = \frac{|\operatorname{Aut}(\Gamma)|}{|\operatorname{Aut}(G)|} e'(\Gamma, G),$ $e'(\Gamma, G) = |\{\gamma \text{ GF on } G : (G, \circ) \simeq \Gamma\}|$

#### SB

HGS

 $(G, \cdot)$ ; there is a bijective correspondence between

- isomorphism classes of skew braces  $(G, \cdot, \circ)$
- classes of gamma functions under "conjugation" by elements of Aut(G):  $\gamma^{\alpha}(g) = \alpha^{-1}\gamma(g^{\alpha^{-1}})\alpha$

Hopf Galois Structures and Skew Braces

What we got and how

## What we got and how

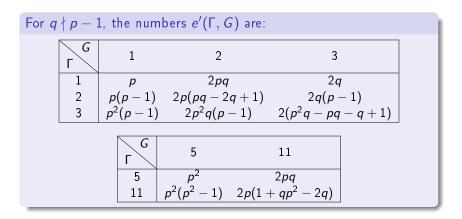
Hopf Galois Structures and Skew Braces

What we got and how Groups of order  $p^2 q$ 

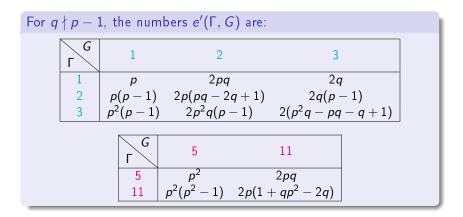
Groups of order  $p^2q$  and their automorphism groups

Туре	Conditions	G	Aut(G)
1		$\mathcal{C}_{p^2}  imes \mathcal{C}_q$	$\mathcal{C}_{p(p-1)}  imes \mathcal{C}_{q-1}$
2	$p \mid q-1$	$\mathcal{C}_q \rtimes_p \mathcal{C}_{p^2}$	$\mathcal{C}_p  imes \operatorname{Hol}(\mathcal{C}_q)$
3	$p^2 \mid q-1$	$\mathcal{C}_q \rtimes_1 \mathcal{C}_{p^2}$	$Hol(\mathcal{C}_q)$
4	$q \mid p-1$	$\mathcal{C}_{p^2} \rtimes \mathcal{C}_q$	$Hol(\mathcal{C}_{p^2})$
5		$\mathcal{C}_p \times \mathcal{C}_p \times \mathcal{C}_q$	$GL(2,p) \times \mathcal{C}_{q-1}$
6	$q \mid p-1$	$\mathcal{C}_p  imes (\mathcal{C}_p  times \mathcal{C}_q)$	$\mathcal{C}_{p-1} imes Hol(\mathcal{C}_p)$
7	$q \mid p-1$	$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{\mathcal{S}} \mathcal{C}_q$	$Hol(\mathcal{C}_p  imes \mathcal{C}_p)$
8	$3 < q \mid p-1$	$(\mathcal{C}_p  imes \mathcal{C}_p) \rtimes_{D0} \mathcal{C}_q$	$Hol(\mathcal{C}_p)  imes Hol(\mathcal{C}_p)$
9	$2 < q \mid p-1$	$(\mathcal{C}_p  imes \mathcal{C}_p) \rtimes_{D1} \mathcal{C}_q$	$(\operatorname{Hol}(\mathcal{C}_p) imes\operatorname{Hol}(\mathcal{C}_p)) times\mathcal{C}_2$
10	$2 < q \mid p+1$	$(\mathcal{C}_p  imes \mathcal{C}_p)  times_{\mathcal{C}} \mathcal{C}_q$	$(\mathcal{C}_{p}  imes \mathcal{C}_{p})  times (\mathcal{C}_{p^{2}-1}  times \mathcal{C}_{2})$
11	$p \mid q-1$	$(\mathcal{C}_{q} \rtimes \mathcal{C}_{p})  imes \mathcal{C}_{p}$	$Hol(\mathcal{C}_p)  imes Hol(\mathcal{C}_q)$

> L/K Galois of order  $p^2q$ , p > 2 and q distinct primes,  $\Gamma = \text{Gal}(L/K)$ . G group of order  $p^2q$ .



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#### First approach:

#### Theorem (Realizability)

Let G be a group of order  $p^2q$  and  $\gamma$  a GF on G. If p > 2, G and  $(G, \circ)$  have isomorphic Sylow p-subgroups.

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- there always exists a Sylow *p*-subgroup *H* which is  $\gamma(H)$ -invariant;
- this corresponds to have (H, ∘) isomorphic to a regular subgroup of Hol(H);

CS19 : H cyclic  $(p > 2) \Rightarrow$  all regular subgrps of Hol(H) are cyclic;

FCC12 : *H* abelian of rank *m* with m , or <math>m = 2 and  $p = 3 \Rightarrow$  all the abelian subgrps of Hol(*H*) are isomorphic to *H*.

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all the abelian subgrps of Hol(H) are isomorphic to H.

If  $\Gamma$ , G of order  $p^2q$ , p > 2, have non isomorphic Sylow p-subgroups,

 $e(\Gamma, G) = e'(\Gamma, G) = 0.$ 

#### Second approach:

Let G be a group,  $A \leq G$ , and  $\gamma : A \rightarrow Aut(G)$  a function.  $\gamma$  is a relative gamma function (RGF) on A if it satisfies the GFE and A is  $\gamma(A)$ -invariant.

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#### Lemma (Morphisms)

G finite group,  $A \leq G$  and  $\gamma : A \rightarrow Aut(G)$  a function such that A is  $\gamma(A)$ -invariant. Any two of the following conditions imply the third one:

- $\gamma([A, \gamma(A)]) = \{1\}. ([x, \gamma(y)] = x^{-1}x^{\gamma(y)}, x, y \in A)$
- $\gamma: A 
  ightarrow {\sf Aut}({\sf G})$  is a morphism.
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#### Third approach:

#### Proposition (Lifting and restriction)

G finite group,  $A, B \leq G$  such that G = AB.

•  $\gamma$  GF on G and  $B \leq \ker(\gamma) \Rightarrow \gamma(ab) = \gamma(a^{\gamma(b)^{-1}})\gamma(b) = \gamma(a)$ 

$$\Rightarrow \gamma(G) = \gamma(A).$$

If A is γ(A)-invariant, then γ<sub>|A</sub>: A → Aut(G) is a RGF on A and ker(γ) is invariant under {γ'(a)ι(a) : a ∈ A} ≤ Aut(G).
If γ': A → Aut(G) is a RGF such that
γ'(A ∩ B) ≡ 1.

2 B is invariant under  $\{\gamma'(a)\iota(a): a \in A\}$ .

Then  $\gamma(ab) = \gamma'(a)$  is a GF on G, and  $\ker(\gamma) = \ker(\gamma')B$ .

Example:  $p \mid q - 1$ , G of type 1, B q-Sylow. Necessarily  $B \leq \ker(\gamma)$ ; moreover A, the p-Sylow, is characteristic  $\Rightarrow \gamma \leftrightarrow \gamma_{\mid A}$ 

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If A is  $\gamma(A)$ -invariant, then  $\gamma_{|A} : A \to Aut(G)$  is a RGF on A and ker $(\gamma)$  is invariant under  $\{\gamma'(a)\iota(a) : a \in A\} \leq Aut(G)$ . If  $\gamma' : A \to Aut(G)$  is a RGF such that  $\gamma'(A \cap B) \equiv 1$ , B is invariant under  $\{\gamma'(a)\iota(a) : a \in A\}$ . Then  $\gamma(ab) = \gamma'(a)$  is a GF on G, and ker $(\gamma) = ker(\gamma')B$ .

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#### Proposition (RGF on cyclic subgroups)

G finite group,  $A = \langle a \rangle$  a cyclic subgroup of G of order  $p^n$  (p odd). Let  $\eta \in Aut(G)$ . The following are equivalent.

- There is a RGF  $\gamma : A \rightarrow Aut(G)$  such that  $\gamma(a) = \eta$ .
  - A is  $\eta$ -invariant, and
    - ord $(\eta) \mid p^n$ .

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Example:  $p \mid q-1$ , G of type 1, B q-Sylow. Necessarily  $B \leq \ker(\gamma)$ ; moreover A, the p-Sylow, is characteristic  $\Rightarrow \gamma \leftrightarrow \gamma_{\mid A}$ ;

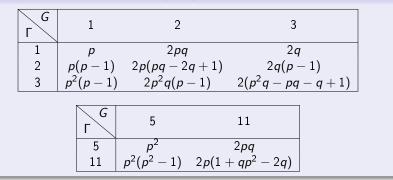
$$\gamma_{|A}: A 
ightarrow \mathsf{Aut}(G) = \mathcal{C}_{p(p-1)} imes \mathcal{C}_{q-1}$$

$$|\mathsf{GF}| = |\mathsf{elements} ext{ of order } | p^2 ext{ in } \mathsf{Aut}(G)| = egin{cases} p^2 ext{ if } p \mid\mid q-1 \ p^3 ext{ if } p^2 \mid q-1 \ p^3 ext{ if } p^2 \mid q-1 \end{cases}$$

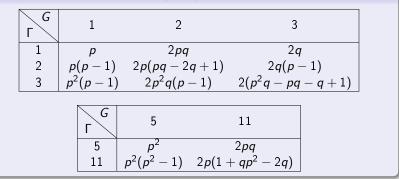


2

For  $q \nmid p - 1$ , the numbers  $e'(\Gamma, G)$  are:

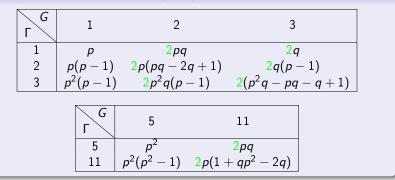


For  $q \nmid p - 1$ , the numbers  $e'(\Gamma, G)$  are:



$$a^{\ominus 1} \circ b \circ a = a^{-\gamma(a)^{-1}\gamma(b)\gamma(a)}b^{\gamma(a)}a$$

For  $q \nmid p - 1$ , the numbers  $e'(\Gamma, G)$  are:



#### Fifth approach: Duality: $\rho(G)^{inv} = \lambda(G)$ , where $inv : x \to x^{-1}$ ,

- ullet the GF associated to the RRR ho(G) is  $\gamma(x)=1$
- the GF associated to the LRR  $\lambda(G)$  is  $\gamma(x) = \iota(x^{-1})$ :

$$y^{\iota(x^{-1})\rho(x)} = xy = y^{\lambda(x)},$$

#### More general:

If  $N \leq Hol(G)$  is a regular subgroup corresponding to  $\gamma$ , then  $N^{inv}$  is another regular subgroup of Hol(G), which corresponds to

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(i) For  $q \nmid p - 1$ :

$\Gamma$	1	2	3
1	p	2p(p-1)	2p(p-1)
2	pq	2p(pq - 2q + 1)	2pq(p - 1)
3	pq	2pq(p - 1)	$2(p^2q - pq - q + 1)$

Г	5	11
5	$p^2$	$2p(p^2 - 1)$ $2p(1 + qp^2 - 2q)$
11	$p^2q$	$2p(1+qp^2-2q)$

(ii) For  $q \nmid p - 1$  and  $q \mid p + 1$ :

Г	5	10
5 10	$p^2$ $p^2$	$\begin{array}{c} p(p-1)(q-1) \\ 2+2p^2(q-3)-p^3+p^4 \end{array}$

(iii) For  $q \mid p - 1$ :

Г	1	4
1	p	2p(q-1) $2(p^2q - 2p^2 + 1)$
- 4	$p^2$	$2(p^2q - 2p^2 + 1)$

If q = 2,

Г	5	6	7
5	$p^2$	2p(p+1)	p(3p + 1)
6	$p^2$	2p(p + 1)	p(3p + 1)
7	$p^2$	$2p^2(p+1)$	2 + p(p+1)(2p-1)

If q = 3,

Г	5	6	7	9
5	$p^2$	4p(p + 1)	2p(3p + 1)	4p(p + 1)
6	p	2p(p + 3)	4p(p + 1)	p(3p + 5)
7	$p^2$	$2p^2(p+1)^2$	$2 + p^2(2p^2 + 3p + 2)$	$p(p+1)^{3}$
9	$p^2(2p-1)$	$4p(p^2 + 1)$	$2(2p^3 + 3p^2 - 2p + 1)$	$2+2p+p^3(p+3)$

If q > 3,

	5	6
5	$p^2$	2p(p+1)(q-1)
6	p	2p(p + 2q - 3)
7	$p^2$	$2p^{2}(p+1)(pq-2p+1)$
$8, G_2$	$p^3$	$4p(p^2 + pq - 3p + 1)$
8, $G_k \not\simeq G_2$	$p^2$	$4p(p^2 + pq - 3p + 1)$
9	$p^2$	$4p(p^2 + pq - 3p + 1)$

$\Gamma$ $G$	7	9
5	p(3p+1)(q-1)	2p(p+1)(q-1)
6	$4(p^2 + pq - 2p)$	p(4q + 3p - 7)
7	$2 + p^2(2p^2 + pq + 2q - 4)$	$p(p+1)(p^2(2q-5)+2p+1)$
$8, G_2$	$2p(p^2q - 4p + pq + 2)$	$p(p^3 + 3p^2 - 14p + 4pq - 6)$
8, $G_k \not\simeq G_2$	$4p(2p^2 - 5p + pq + 2)$	$p(p^3 + 5p^2 - 18p + 4pq + 8)$
9	$2(4p^3 - 9p^2 + 2p^2q + 2p + 1)$	$2 + 4p + p^2(p^2 + 5p + 4q - 16)$

	68 Г	$G \not\simeq G_{\pm 2}$	$G \simeq G_{\pm 2}, q > 5$	$G \simeq G_2, q = 5$
Г	5	4p(p+1)(q-1)	4p(p+1)(q-1)	16p(p + 1)
	6	8p(q + p - 2)	8p(q + p - 2)	8p(p + 3)
	7	$4p^2(p+1)(pq-3p+2)$	$4p^2(p+1)(pq-3p+2)$	$8p^2(p+1)^2$
	8	Table 2	Table 1	$4(1 + p + 3p^2(p + 1))$
L	9	$8p(2p^2 + pq - 5p + 2))$	$4p(3p^2 + 2pq - 8p + 3)$	$16p(2p^3 - 2p + p + 1)$

Table 1: G and I of	type 8, $G \simeq G_k$ for $\kappa = \pm 2$ ,
Г	if $q > 7$ :
$G_2$	$2(1 + 5p + 4p^2q - 17p^2 + 7p^3)$
$G_3, G_{\frac{3}{2}}$	$2(7p + 4p^2q - 18p^2 + 7p^3)$
G_2 2	$2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$
$G_{s} \not\simeq G_{2}, G_{3}, G_{\frac{3}{2}}, G_{-2}$	$8(2p + p^2q - 5p^2 + 2p^3)$
Г	if $q = 7$ :
$G_2$	$2(1 + 5p + 11p^2 + 7p^3)$
$G_3$	$2(1 + 4p + 13p^2 + 6p^3)$

Table 1: G and  $\Gamma$  of type 8,  $G \simeq G_k$  for  $k = \pm 2$ .

Table 2: G and  $\Gamma$  of type 8,  $G \simeq G_k \not\simeq G_{\pm 2}$ 

Г	if either k or $k^{-1}$ is a solution of $x^2 - x - 1 = 0$ :
$G_k, G_{1-k}$	$2(1 + 5p + 4p^2q - 17p^2 + 7p^3)$
$G_{1+k}$	$4(3p + 2p^2q - 8p^2 + 3p^3)$
$G_s \not\simeq G_k, G_{1+k}, G_{1-k}$	$8(2p + p^2q - 5p^2 + 2p^3)$
Г	if k and $k^{-1}$ are the solutions of $x^2 + x + 1 = 0$ :
G <sub>k</sub>	$2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$
$G_{1-k}, G_{1-k^{-1}}$	$2(7p + 4p^2q - 18p^2 + 7p^3)$
$G_{1+k}$	$2(1 + 4p + 4p^2q - 15p^2 + 6p^3)$
$G_{s} \not\simeq G_{k}, G_{1+k}, G_{1-k}, G_{1-k^{-1}}$	$8(2p + p^2q - 5p^2 + 2p^3)$
Г	if k and $k^{-1}$ are the solutions of $x^2 - x + 1 = 0$ :
$G_{-k}$	$2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$
$G_{1+k}, G_{1+k^{-1}}$	$2(7p + 4p^2q - 18p^2 + 7p^3)$
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$G_s \not\simeq G_{-k}, G_{1-k}, G_{1+k}, G_{1+k^{-1}}$	$8(2p + p^2q - 5p^2 + 2p^3)$
Г	if k and $k^{-1}$ are the solutions of $x^2 + 1 = 0$ :
$G_k$	$4(1 + 2p + 2p^2q - 9p^2 + 4p^3)$
$G_{1+k}, G_{1-k}$	$4(3p + 2p^2q - 8p^2 + 3p^3)$
$G_s \not\simeq G_k, G_{1+k}, G_{1-k}$	$8(2p + p^2q - 5p^2 + 2p^3)$
Г	if $k^2 \neq \pm k \pm 1, -1$ :
$G_k, G_{-k}$	$2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$
$G_{1+k}, G_{1+k^{-1}}, G_{1-k}, G_{1-k^{-1}}$	$2(7p + 4p^2q - 18p^2 + 7p^3)$
$G_s \not\simeq G_{\pm k}, G_{1\pm k}, G_{1\pm k^{-1}}$	$8(2p + p^2q - 5p^2 + 2p^3)$

## Thank you for the attention!